# Semidefinite Bounds for the Stability Number of a Graph via Sums of Squares of Polynomials

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**Abstract.** Lovász and Schrijver [9] have constructed semidefinite relaxations for the stable set polytope of a graph $G = (V, E)$ by a sequence of lift-and-project operations; their procedure finds the stable set polytope in at most $\alpha(G)$ steps, where $\alpha(G)$ is the stability number of $G$. Two other hierarchies of semidefinite bounds for the stability number have been proposed by Lasserre [4],[5] and by de Klerk and Pasechnik [3], which are based on relaxing nonnegativity of a polynomial by requiring the existence of a sum of squares decomposition. The hierarchy of Lasserre is known to converge in $\alpha(G)$ steps as it refines the hierarchy of Lovász and Schrijver, and de Klerk and Pasechnik conjecture that their hierarchy also finds the stability number after $\alpha(G)$ steps. We prove this conjecture for graphs with stability number at most 8 and we show that the hierarchy of Lasserre refines the hierarchy of de Klerk and Pasechnik.

## 1 Introduction

Semidefinite programming plays an essential role for constructing good relaxations for hard combinatorial optimization problems, in particular, for the maximum stable set problem which will be considered in the present paper. Lovász [8] introduced the theta number $\vartheta(G)$ as an upper bound for the stability number $\alpha(G)$ of a graph $G$; $\vartheta(G)$ can be computed efficiently (to any arbitrary precision) using semidefinite programming and it coincides with $\alpha(G)$ when $G$ is a perfect graph. Lovász and Schrijver [9] construct a hierarchy of semidefinite relaxations for the stable set polytope of $G$ by a sequence of lift-and-project operations; their procedure is finite and it finds the stable set polytope in at most $\alpha(G)$ steps.

Two other hierarchies of semidefinite bounds for the stability number have been proposed by Lasserre [4, 5] and by de Klerk and Pasechnik [3]. They are based on the following paradigm: While testing nonnegativity of a polynomial is a hard problem, one can test efficiently whether a polynomial can be written as a sum of squares of polynomials via semidefinite programming. As was already proved by Hilbert in 1888 not every nonnegative multivariate polynomial can be written as a sum of squares (see Reznick [14] for a nice survey on this

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topic). However, some representation theorems have been proved ensuring the existence of certain sums of squares decompositions under some assumption, like positivity of the polynomial on a compact basic closed semi-algebraic set (see, e.g., [18] for an exposition of such results). An early such result is due to Pólya [13] who showed that, if $p(x)$ is a homogeneous polynomial which is positive on $\mathbb{R}^n_+ \setminus \{0\}$, then $(\sum_{i=1}^n x_i)^r p(x)$ has only nonnegative coefficients (and thus $(\sum_{i=1}^n x_i^2)^r p(x_1^2, \ldots, x_n^2)$ is a sum of squares) for some sufficiently large integer $r$.

The starting point for Lasserre's construction is that the stability number $\alpha(G)$ of a graph $G = (V, E)$ can be expressed as the smallest scalar $t$ for which the polynomial $t - \sum_{i \in V} x_i$ is nonnegative on the set $\{x \in \mathbb{R}^V \mid x_i x_j = 0 \ (ij \in E),\ x_i^2 = x_i \ (i \in V)\}$. Requiring the weaker condition that the polynomial $t - \sum_{i \in V} x_i$ can be written as a sum of squares modulo the ideal generated by $x_i x_j$ $(ij \in E)$ and $x_i^2 - x_i$ $(i \in V)$ with given degree bounds, yields a hierarchy of semidefinite upper bounds for $\alpha(G)$. The dual approach (in terms of moment matrices) yields the hierarchy of Lasserre [4, 5] of semidefinite relaxations for the stable set polytope. This hierarchy refines the hierarchy of Lovász and Schrijver (see [6]) and thus it also finds the stable set polytope in $\alpha(G)$ steps.

By a result of Motzkin and Straus [11], one may alternatively express $\alpha(G)$ as the smallest scalar $t$ for which the matrix $M := t(I + A_G) - J$ (with entries $t - 1$ on the diagonal and at positions corresponding to edges and $-1$ elsewhere) is copositive, meaning that the polynomial $p_M(x) := \sum_{i,j \in V} x_i^2 x_j^2 M_{ij}$ is nonnegative on $\mathbb{R}^n$. Following Parrilo [12], de Klerk and Pasechnik [3] propose to relax the nonnegativity condition on $p_M(x)$ and to require instead that $(\sum_{i \in V} x_i^2)^r p_M(x)$ be a sum of squares for some integer $r \geq 0$. The convergence of these bounds to $\alpha(G)$ is guaranteed by the above mentioned result of Pólya. The first bound in the hierarchy coincides with the strengthening $\vartheta'(G)$ of the theta number introduced by McEliece, Rodemich and Rumsey [10] and Schrijver [16]. It is however not clear how the next bounds relate to the bounds provided by the construction of Lasserre. It is conjectured in [3] that the stability number is found after $\alpha(G)$ steps. In this paper we study this conjecture and develop a proof technique which enables us to show that the conjecture holds for graphs with stability number at most 8. Moreover, we show that the hierarchy of bounds of Lasserre (enhanced by adding some nonnegativity constraint) refines the hierarchy of bounds of de Klerk and Pasechnik, answering another open question of [3].

The paper is organized as follows. In Section 2, we first recall some definitions and results related to the hierarchies of bounds of Lasserre and of de Klerk and Pasechnik. Then we introduce a dual formulation for the latter bounds, which will enable us to compare the two hierarchies of bounds, and we present our main results. The proofs are delayed till Section 3, where we prove the conjecture for graphs with stability number at most 8, and till Section 4, where we prove the relation between the two hierarchies.

Throughout, $G = (V, E)$ denotes a graph with node set $V = \{1, \ldots, n\}$. Let $\alpha(G)$ denote its stability number, i.e., the largest cardinality of a stable set in $G$, and let $A_G$ denote the adjacency matrix of $G$, i.e., $A_G$ is the 0/1 matrix indexed by $V$ whose $(i,j)$-th entry is 1 when $i,j \in V$ are connected by an edge. All matrices are assumed to be symmetric and $I$, $J$, $e$, $e_i$ $(i = 1,\ldots,n)$ denote, respectively, the identity matrix, the all-ones matrix, the all-ones vector, and the standard unit vectors of suitable sizes. A matrix $M$ is *copositive* if $x^T M x \geq 0$ for all $x \in \mathbb{R}^n_+$ and $\mathcal{C}_n$ denotes the *copositive cone*, consisting of the $n \times n$ copositive matrices. For a symmetric matrix $M$, we write $M \succeq 0$ if $M$ is positive semidefinite. For a sequence $\beta \in \mathbb{Z}^n_+$, we set $|\beta| := \sum_{i=1}^n \beta_i$, $\beta! := \beta_1! \cdots \beta_n!$, $S(\beta) := \{i \mid \beta_i \neq 0\}$, and $S_{odd}(\beta) := \{i \mid \beta_i \text{ is odd}\}$. One says that $\beta$ is even when $S_{odd}(\beta) = \emptyset$. Finally, set $I(n,r) := \{\beta \in \mathbb{Z}^n_+ \mid |\beta| = r\}$ and $\mathcal{P}_r(V) := \{I \subseteq V \mid |I| \leq r\}$.

## 2 Semidefinite Bounds for the Stability Number

### 2.1 The Semidefinite Bounds of Lasserre

Given an integer $r \geq 1$ and a vector $x = (x_I)_{I \in \mathcal{P}_{2r}(V)}$, consider the matrix:

$$M_r(x) := (x_{I \cup J})_{I,J \in \mathcal{P}_r(V)}$$

known as the moment matrix of $x$ of order $r$. By setting:

$$las^{(r)}(G) := \max \textstyle\sum_{i \in V} x_i \ \text{ s.t. } \begin{array}{l} M_r(x) \succeq 0,\ x_I \geq 0\ (I \subseteq V, |I| = r+1), \\ x_\emptyset = 1,\ x_{ij} = 0\ (ij \in E) \end{array} \tag{1}$$

one obtains a hierarchy of semidefinite bounds for the stability number, known as Lasserre's hierarchy [5, 6]. Indeed, if $S$ is a stable set, the vector $x \in \mathbb{R}^{\mathcal{P}_{2r}(V)}$ with $x_I = 1$ if $I \subseteq S$ and $x_I = 0$ otherwise, is feasible for (1) with objective value $|S|$, showing $\alpha(G) \leq las^{(r)}(G)$. We note that $las^{(1)}(G) = \vartheta'(G)$. For fixed $r$, the parameter $las^{(r)}(G)$ can be computed in polynomial time (to an arbitrary precision) since the semidefinite program (1) involves matrices of size $O(n^r)$ with $O(n^{2r})$ variables.

Equality $\alpha(G) = las^{(r)}(G)$ holds for $r \geq \alpha(G)$. This result remains valid if we remove the nonnegativity constraint: $x_I \geq 0$ $(|I| = r+1)$ in (1) ([6]). However, with this nonnegativity condition, we will be able to compare the hierarchies of Lasserre and de Klerk and Pasechnik (see Theorem 3 below).

### 2.2 The Semidefinite Bounds of de Klerk and Pasechnik

The starting point in [3] is the following formulation for $\alpha(G)$ found by Motzkin and Straus [11]:

$$\frac{1}{\alpha(G)} = \min\ x^T(I + A_G)x \ \text{ subject to } x \geq 0,\ \sum_{i=1}^n x_i = 1.$$

In other words,

$$\alpha(G) = \min\ t \ \text{ subject to } t(I + A_G) - J \in \mathcal{C}_n. \tag{2}$$

Therefore, upper bounds for $\alpha(G)$ can be obtained by replacing in program (2) the copositive cone $\mathcal{C}_n$ by a smaller subcone of it. Following [12], given an integer $r \geq 0$, $\mathcal{K}_n^{(r)}$ is the cone of $n \times n$ matrices $M$ for which the polynomial

$$p_M^{(r)}(x) := \left(\sum_{i=1}^n x_i^2\right)^r \left(\sum_{i,j=1}^n M_{ij} x_i^2 x_j^2\right) \tag{3}$$

can be written as a sum of squares of polynomials. Parrilo [12] shows that

$$\mathcal{K}_n^{(0)} = \{P + N \mid P \succeq 0, N \geq 0\}. \tag{4}$$

A characterization of $\mathcal{K}_n^{(1)}$ can be found in [12, 1]. Obviously, $\mathcal{K}_n^{(r)} \subseteq \mathcal{K}_n^{(r+1)} \subseteq \ldots \subseteq \mathcal{C}_n$. The result of Pólya mentioned in the Introduction shows that the interior of the cone $\mathcal{C}_n$ is equal to $\bigcup_{r \geq 0} \mathcal{K}_n^{(r)}$. Setting

$$\vartheta^{(r)}(G) := \min \ t \ \text{ subject to } \ t(I + A_G) - J \in \mathcal{K}_n^{(r)}, \tag{5}$$

one obtains a hierarchy of upper bounds for $\alpha(G)$. The first bound $\vartheta^{(0)}(G)$ is equal to

$$\vartheta'(G) = \max \ \text{Tr}(JX) \ \text{ s.t. } \text{Tr}(X) = 1, \ X_{ij} = 0 \ (ij \in E), \ X \succeq 0, \ X \geq 0 \tag{6}$$

(see [3]). Thus, $\vartheta^{(0)}(G) \leq \vartheta(G)$, since program (6) without the nonnegativity condition is a formulation of the theta number.

The problem of finding a sum of squares decomposition for a polynomial of degree $2d$ can be formulated as a semidefinite program involving matrices of size $O(n^d)$ and $O(n^{2d})$ variables (see, e.g., [12]). Therefore, for fixed $r$, program (5) can be reformulated as a semidefinite program of polynomial size and thus $\vartheta^{(r)}(G)$ can be computed in polynomial time (to any precision).

De Klerk and Pasechnik [3] show that

$$\alpha(G) = \lfloor \vartheta^{(r)}(G) \rfloor \text{ for } r \geq \alpha(G)^2.$$

Indeed the matrix $M := \alpha(1+\epsilon)(I + A_G) - J$ with $\alpha = \alpha(G)$ and $\epsilon = \frac{\alpha - 1}{\alpha^2 - \alpha + 1}$, belongs to the cone $\mathcal{K}_n^{(r)}$ since all the coefficients of the polynomial $p_M^{(r)}(x)$ are nonnegative; this implies that $\alpha(G) \leq \vartheta^{(r)}(G) \leq \alpha(G)(1+\epsilon) < \alpha(G) + 1$.

Let us observe that, for the matrix $M := \alpha(I + A_G) - J$, the polynomial $p_M^{(r)}(x)$ has a negative coefficient for any $r \geq 0$ when $\alpha = \alpha(G) \geq 2$. To see it, recall from [1] that

$$p_M^{(r)}(x) = \sum_{\beta \in I(n, r+2)} \frac{r!}{\beta!} c_\beta x^{2\beta}, \text{ where } c_\beta := \beta^T M \beta - \beta^T \text{diag}(M). \tag{7}$$

If $S(\beta)$ is a stable set, then $c_\beta = \alpha \sum_i \beta_i(\beta_i - 1) - (r+1)(r+2)$. Write $r + 2 = q\alpha + s$ with $q, s \in \mathbb{Z}_+$, $0 \leq s < \alpha$; then $c_\beta < 0$ for $\beta = (q+1, \ldots, q+1, q, \ldots, q, 0, \ldots 0)$

with $s$ entries equal to $q+1$, $\alpha - s$ entries equal to $q$, and $S(\beta)$ being a stable set.

It is also shown in [3] that

$$\vartheta^{(1)}(G) \leq 1 + \max_{i \in V} \vartheta^{(0)}(G \backslash i^{\perp}) \tag{8}$$

where, for $i \in V$, $G\backslash i^{\perp}$ is the graph obtained from $G$ by deleting $i$ and its neighbours. Therefore, $\vartheta^{(1)}(G) = \alpha(G)$ when $\alpha(G) \leq 2$. More generally, de Klerk and Pasechnik [3] conjecture:

**Conjecture 1.** $\vartheta^{(r)}(G) = \alpha(G)$ for $r \geq \alpha(G) - 1$.

### 2.3 Dual Formulation

Using conic duality, the bound $\vartheta^{(r)}(G)$ from (5) can be reformulated as

$$\vartheta^{(r)}(G) = \max \ \mathrm{Tr}(JX) \ \text{ subject to } \mathrm{Tr}((I + A_G)X) = 1, \ X \in (\mathcal{K}_n^{(r)})^*. \tag{9}$$

As the programs (5) and (9) are strictly feasible, there is no duality gap and the optima in (5) and (9) are indeed attained ([3]). For $r = 0$, it follows from (4) that $(\mathcal{K}_n^{(0)})^*$ is the cone of completely positive (i.e., positive semidefinite and nonnegative) matrices. For $r \geq 1$, one can give an explicit description of the dual cone $(\mathcal{K}_n^{(r)})^*$ in terms of moment matrices.

**Definition 1.** *Let $y = (y_\delta)_{\delta \in I(n,2r+4)}$ be given.*

*(i) Define the matrix $N_{r+2}(y)$ indexed by $I(n, r+2)$, whose $(\beta, \beta')$-th entry is equal to $y_{\beta+\beta'}$, for $\beta, \beta' \in I(n, r+2)$.*
*(ii) For $\gamma \in I(n, r)$, $N^\gamma(y)$ denotes the principal submatrix of $N_{r+2}(y)$ indexed by $\gamma + 2e_1, \ldots, \gamma + 2e_n$; that is, $N^\gamma(y)$ is the $n \times n$ matrix with $(i, j)$-th entry $y_{2\gamma+2e_i+2e_j}$, for $i, j = 1, \ldots, n$.*
*(iii) Define the $n \times n$ matrix*

$$C(y) := \sum_{\gamma \in I(n,r)} \frac{r!}{\gamma!} N^\gamma(y). \tag{10}$$

**Definition 2.** *Define the cone*

$$\mathcal{C}_n^{(r)} := \{Z \in \mathbb{R}^{n \times n} \mid Z = C(y) \ \text{ for some } y \in \mathbb{R}^{I(n,2r+4)} \text{ with } N_{r+2}(y) \succeq 0\}.$$

*As the matrix $C(y)$ in (10) involves only entries of $y$ indexed by even sequences, one can assume w.l.o.g. in the definition of the cone $\mathcal{C}_n^{(r)}$ that $y_\delta = 0$ whenever $\delta$ has an odd component.*

**Lemma 1.** *The cones $\mathcal{K}_n^{(r)}$ and $\mathcal{C}_n^{(r)}$ are dual of each other; i.e., $\mathcal{C}_n^{(r)} = (\mathcal{K}_n^{(r)})^*$ and $\mathcal{K}_n^{(r)} = (\mathcal{C}_n^{(r)})^*$.*

*Proof.* For a polynomial $p(x) = \sum_\delta p_\delta x^\delta$, let $p := (p_\delta)_\delta$ denote the vector of its coefficients. The following can be easily verified:

$$y^T v = u^T(N_{r+2}(y))u \ \text{ for } y \in \mathbb{R}^{I(n,2r+4)},\ u \in \mathbb{R}^{I(n,r+2)},\ v(x) := u(x)^2. \quad (11)$$

Consider the cones $\mathcal{C} := \{y \in \mathbb{R}^{I(n,2r+4)} \mid N_{r+2}(y) \succeq 0\}$, $\mathcal{D} := \{p \in \mathbb{R}^{I(n,2r+4)} \mid$ the polynomial $p(x)$ is a sum of squares$\}$. Then, $\mathcal{C} = \mathcal{D}^*$ as a direct application of (11), which implies $\mathcal{D} = \mathcal{C}^*$ since $\mathcal{D}$ is a closed cone (see [15]). Using (7), one can also easily verify that

$$Tr(MC(y)) = y^T(p_M^{(r)}) \ \text{ for } y \in \mathbb{R}^{I(n,2r+4)},\ M \text{ symmetric } n \times n \text{ matrix} \quad (12)$$

where $p_M^{(r)}(x)$ is the polynomial from (3). We can now prove the lemma. As $\mathcal{C}_n^{(r)}$ is a closed cone, it suffices to show: $\mathcal{K}_n^{(r)} = (\mathcal{C}_n^{(r)})^*$. The inclusion $\mathcal{K}_n^{(r)} \subseteq (\mathcal{C}_n^{(r)})^*$ follows using (11) and (12). Conversely, let $M \in (\mathcal{C}_n^{(r)})^*$. Then, by (12), $y^T(p_M^{(r)}) \geq 0$ for all $y \in \mathcal{C}$; that is, $p_M^{(r)} \in \mathcal{C}^* = \mathcal{D}$, showing that $M \in \mathcal{K}_n^{(r)}$. □

Consider the program

$$\tilde{\vartheta}^{(r)}(G) := \max\ \text{Tr}(X) \ \text{ s.t. } \begin{array}{l} X \in \mathcal{C}_n^{(r)} = (\mathcal{K}_n^{(r)})^*,\ \text{Tr}(A_G X) = 0, \\ X - \text{diag}(X)\text{diag}(X)^T \succeq 0. \end{array} \quad (13)$$

Then,

$$\alpha(G) \leq \tilde{\vartheta}^{(r)}(G) \leq \vartheta^{(r)}(G). \quad (14)$$

Indeed, if $X$ is feasible for (13), then $X' := \frac{X}{Tr(X)}$ is feasible for (9) with $Tr(JX') \geq Tr(X)$, which shows $\tilde{\vartheta}^{(r)}(G) \leq \vartheta^{(r)}(G)$. Given a stable set $S$ with incidence vector $x := \chi^S$, define the vector $y \in \mathbb{R}^{I(n,2r+4)}$ with $y_\delta = \frac{1}{|S|^r}$ if $\delta$ is even and $S(\delta) \subseteq S$, and $y_\delta = 0$ otherwise. Then, $N_{r+2}(y) \succeq 0$; $X := C(y) = xx^T$ is feasible for (13) with $Tr(X) = |S|$, which shows $\alpha(G) \leq \tilde{\vartheta}^{(r)}(G)$.

### 2.4 The Main Results

Our main results are the following:

**Theorem 1.** *For a graph $G$ and a positive integer $r \leq \min(\alpha(G) - 1, 6)$,*

$$\vartheta^{(r)}(G) \leq r + \max_{S \subseteq V \text{ stable},\ |S| = r} \vartheta^{(0)}(G \backslash S^\perp), \quad (15)$$

*where $S^\perp$ denotes the set of nodes that belong to $S$ or are adjacent to a node in $S$.*

**Theorem 2.** *Conjecture 1 holds for $\alpha(G) \leq 8$; that is,*

$$\vartheta^{(\alpha(G)-1)}(G) = \alpha(G) \quad \text{if } \alpha(G) \leq 8.$$

**Theorem 3.** *For $r \geq 1$, the parameters from (1),(9) and (13) satisfy:*

$$las^{(r)}(G) \leq \tilde{\vartheta}^{(r-1)}(G) \leq \vartheta^{(r-1)}(G). \quad (16)$$

Theorem 2 follows directly from Theorem 1 when $\alpha(G) \leq 7$ and from its proof technique when $\alpha(G) = 8$. Our proof technique does not apply to the case when $\alpha(G) \geq 9$. It is quite more complicated than the proof of convergence in $\alpha(G)$ steps for the Lovász-Schrijver and the Lasserre semidefinite hierarchies. One of the main difficulties (as pointed out later in the proof) comes from the fact that, for $r \geq 1$, the cone $\mathcal{K}_n^{(r)}$ is not invariant under some simple matrix operations, like extending a matrix by adding a zero row and column to it, or rescaling it by positive multipliers (which obviously preserve copositivity and positive semidefiniteness). For instance, when $G$ is a circuit of length 5, the matrix $M := 2(I + A_G) - J$ belongs to $\mathcal{K}_5^{(1)}$, but adding a zero row and column yields a matrix that does not belong to $\mathcal{K}_6^{(1)}$. We thank E. de Klerk for communicating this example to us.

As Theorem 3 shows, the bound $las^{(r)}(G)$ is at least as good as $\tilde{\vartheta}^{(r-1)}(G)$. There exist in fact graphs for which strict inequality: $las^{(2)}(G) < \tilde{\vartheta}^{(1)}(G)$ holds. For this, given integers $2 \leq d \leq n$, consider the graph $G(n,d)$ with node set $\mathcal{P}(V)$ ($|V| = n$) where $I, J \in \mathcal{P}(V)$ are connected by an edge if $|I \Delta J| \in \{1, \ldots, d-1\}$. Then $\alpha(G(n,d))$ is the maximum cardinality of a binary code of word length $n$ with minimum distance $d$. Delsarte [2] introduced a linear programming bound which coincides with the parameter $\vartheta'(G(n,d))$ ([16]). Schrijver [17] introduced a stronger semidefinite bound which roughly[1] lies between the bounds $las^{(1)}(G(n,d))$ and $las^{(2)}(G(n,d))$ ([7]). While $G(n,d)$ has $2^n$ vertices, Schrijver's bound can be computed via a semidefinite program of size $O(n^3)$ (using a block-diagonalization of the underlying Terwiliger algebra). It turns out that the same algebraic property holds for the bound $\vartheta^{(1)}(G(n,d))$; thus we could compute this bound as well as Schrijver's bound for the parameters $(n,d) = (17,4), (17,6), (17,8)$, and we found:

$$las^{(2)}(G(17,4)) \leq 3276 < 3607 \leq \vartheta^{(1)}(G(17,4))$$
$$las^{(2)}(G(17,6)) \leq 352 < 395 \leq \vartheta^{(1)}(G(17,6))$$
$$las^{(2)}(G(17,8)) \leq 41 < 42 \leq \vartheta^{(1)}(G(17,8)).$$

## 3 Proofs of Theorems 1 and 2

Let $G = (V, E)$ be a graph with stability number $\alpha(G)$, $V = \{1, \ldots, n\}$ and $1 \leq r \leq \alpha(G) - 1$ an integer. Set

$$t := r + \max_{S \subseteq V stable, |S| = r} \vartheta^{(0)}(G \backslash S^{\perp}).$$

Then, $t \geq r + 1$. By assumption,

$$(t - r)(I + A_{G \backslash S^{\perp}}) - J \in \mathcal{K}^{(0)}_{n - |S^{\perp}|} \text{ for any stable set } S \text{ in } G \text{ of size } r. \quad (17)$$

[1] Indeed, the formulation of Schrijver's bound has an additional constraint, namely, $x_{ijk} \leq x_{ij}$ for all $i, j, k \in V$, which does not appear in the definition of the bound $las^{(r)}(G)$ used in the present paper.

In order to prove Theorem 1, we have to show that, for $1 \le r \le \min(\alpha(G)-1, 6)$,

$$M := t(I + A_G) - J \in \mathcal{K}_n^{(r)}. \tag{18}$$

We need some notation. For two nodes $u, v \in V$, write $u \simeq v$ if $u = v$ or $uv \in E$, and $u \not\simeq v$ otherwise. For $x \in \mathbb{R}^n$, set $v(x) := (x_i^2)_{i=1}^n$. Let $B$ be an $m \times n$ matrix. We say that $B$ is a $q \times s$ *block matrix* if the set $\{1, \dots, m\}$ indexing its rows can be partitioned into $Q_1 \cup \ldots \cup Q_q$ and the set $\{1, \dots, n\}$ indexing its columns can be partitioned into $S_1 \cup \ldots \cup S_s$ in such a way that, for any $h \in \{1, \dots, q\}$, $h' \in \{1, \dots, s\}$, the entries $B_{ij}$ for $i \in Q_h, j \in S_{h'}$ are all equal to the same value, say $\tilde{b}_{hh'}$. In other words, $B$ is obtained from the matrix $\tilde{B} := (\tilde{b}_{hh'})_{\substack{h \in \{1,\dots,q\} \\ h' \in \{1,\dots,s\}}}$ by suitably duplicating rows and columns. Obviously, $B \succeq 0$ if and only if $\tilde{B} \succeq 0$. We call $\tilde{B}$ the *skeleton* of the block matrix $B$.

The following observation plays a central role in the proof.

**Lemma 2.** *Let $X(i)$ $(i \in V)$ be symmetric matrices satisfying the condition:*

$$X(i)_{jk} + X(j)_{ik} + X(k)_{ij} \ge 0 \quad \textit{for all } i, j, k \in V, \tag{19}$$

*then the polynomial $\sum_{i \in V} x_i^2 v(x)^T X(i) v(x) = \sum_{i,j,k \in V} x_i^2 x_j^2 x_k^2 X(i)_{jk}$ is a sum of squares.*

*Proof.* The polynomial $\sum_{i,j,k \in V} x_i^2 x_j^2 x_k^2 X(i)_{jk}$ is equal to

$$\begin{aligned} &\sum_{\substack{(i,j,k) \in V^3 \\ i \ne j \ne k \ne i}} x_i^2 x_j^2 x_k^2 [X(i)_{jk} + X(i)_{jk} + X(i)_{jk}] \\ &+ \sum_{\substack{(i,j) \in V^2 \\ i \ne j}} x_i^2 x_j^4 [X(i)_{jj} + 2X(j)_{ij}] + \sum_{i \in V} x_i^6 X(i)_{ii}, \end{aligned}$$

which is a sum of squares, since all coefficients are nonnegative by (19). □

Our strategy will be to construct matrices $X(\{i_1, ..., i_k\}, i)$ $(i \in V)$ satisfying (19) when $\{i_1, ..., i_k\}$ is a stable set of size $k \le r$. We will use them to recursively decompose $M$ into $M - X(i_1) - X(i_1, i_2) - \ldots - X(i_1, \dots, i_k)$ in such way that at the last level $k = r$ we obtain matrices in $\mathcal{K}_n^{(0)}$.

### 3.1 Defining Sets of Matrices Satisfying the Linear Condition (19)

Let $S$ be a stable set of cardinality $k$, $0 \le k \le r$. We define a set of matrices $X(S, i)$ (for $i \in V$) indexed by $V$ that satisfy the condition (19). Set $m_0 := 1$ and $m_k := \frac{t^k}{(t-1)\cdots(t-k)}$ for $k = 1, \dots, r$. (Then, $t \ge r + 1 > k$.)

For $i \in S^\perp$, $X(S, i)$ is the symmetric matrix whose entry at position $(u, v)$ is defined as follows:

$$m_k \text{ times } \begin{cases} 0 & \text{if } u \text{ or } v \in S^\perp \\ t - k - 1 & \text{if } u, v \in V \setminus S^\perp \text{ and } u \simeq v \\ -1 & \text{if } u, v \in V \setminus S^\perp \text{ and } u \not\simeq v. \end{cases}$$

For $i \notin S^\perp$, $X(S,i)$ is the symmetric matrix whose entry at position $(u,v)$ is defined as follows:

$$m_k \text{ times } \begin{cases} 0 & \text{if } u,v \in S^\perp \\ -\frac{t-k-1}{2} & \text{if } u \in S^\perp, v \in i^\perp \setminus S^\perp \\ \frac{1}{2} & \text{if } u \in S^\perp,\ v \in V \setminus (S^\perp \cup i^\perp) \\ 0 & \text{if } u,v \in i^\perp \setminus S^\perp \text{ and } u \simeq v \\ -(t-k) & \text{if } u,v \in i^\perp \setminus S^\perp \text{ and } u \not\simeq v \\ t-\frac{k}{2} & \text{if } u \in i^\perp \setminus S^\perp,\ v \in V \setminus (S^\perp \cup i^\perp) \text{ and } u \simeq v \\ \frac{k}{2} & \text{if } u \in i^\perp \setminus S^\perp,\ v \in V \setminus (S^\perp \cup i^\perp) \text{ and } u \not\simeq v \\ -k & \text{if } u,v \in V \setminus (S^\perp \cup i^\perp) \text{ and } u \simeq v \\ 0 & \text{if } u,v \in V \setminus (S^\perp \cup i^\perp) \text{ and } u \not\simeq v. \end{cases}$$

If $S = \{i_1, \ldots, i_k\}$, we also denote $X(S,i)$ as $X(i_1, \ldots, i_k, i)$. When $S = \emptyset$, we set $X(\emptyset, i) =: X(i)$. Given an ordering $(S) = (i_1, \ldots, i_k)$ of the elements of $S$, define the matrix

$$M((S)) := M - X(i_1) - X(i_1, i_2) - \ldots - X(i_1, \ldots, i_k). \tag{20}$$

**Lemma 3.** *Given a stable set $S$ of size $0 \le k \le r$, the matrices $X(S,i)$ $(i \in V)$ satisfy (19).*

*Proof.* Direct verification. □

### 3.2 The Role of the Matrices $X(S,i)$ and $M((S))$ in the Proof

Our objective is to prove that the matrix $M$ from (18) belongs to the cone $\mathcal{K}_n^{(r)}$, i.e., that the polynomial $p_M^{(r)}(x) = \sigma(x)^r v(x)^T M v(x)$ is a sum of squares, setting $\sigma(x) := \sum_{i=1}^n x_i^2$. The basic idea is to decompose $p_M^{(r)}(x)$ as

$$\sigma(x)^{r-1} \sum_{i=1}^n x_i^2 v(x)^T (M - X(i)) v(x) + \sigma(x)^{r-1} \sum_{i=1}^n x_i^2 v(x)^T X(i) v(x). \tag{21}$$

The second sum is a sum of squares by Lemmas 2 and 3. Each matrix $M - X(i)$ can be written as

$$M - X(i) = \begin{array}{c} \\ i^\perp \\ V \setminus i^\perp \end{array} \begin{array}{c} \begin{array}{cc} \quad i^\perp \quad & \quad V \setminus i^\perp \end{array} \\ \begin{pmatrix} (t-1)J & -J \\ -J & t(I + A_{G \setminus i^\perp}) - J \end{pmatrix} \end{array} \tag{22}$$

$$= \frac{t}{t-1} \begin{pmatrix} 0 & 0 \\ 0 & (t-1)(I + A_{G \setminus i^\perp}) - J \end{pmatrix} + \begin{pmatrix} (t-1)J & -J \\ -J & \frac{1}{t-1} J \end{pmatrix}. \tag{23}$$

When $r = 1$, (22),(23) together with assumption (17) imply that $M - X(i) \in \mathcal{K}_n^{(0)}$ and thus $p_M^{(1)}(x)$ is a sum of squares; therefore, (8) holds. Assume now $r \ge 2$. The last matrix in (23) is positive semidefinite. Suppose our assumption would be that $(t-1)(I - A_{G\setminus i^\perp}) - J \in \mathcal{K}^{(r-1)}_{n-|i^\perp|}$, then it would be tempting to conclude from (22) and (23) that $M - X(i) \in \mathcal{K}^{(r-1)}_n$ (which would then imply that $M \in \mathcal{K}^{(r)}_n$ and thus conclude the proof). This would be correct if we would work with cones of matrices which are closed under adding a zero row and column, but this is not the case for the cones $\mathcal{K}^{(r)}$ and thus this argument does not work. To go around this difficulty, we further decompose the first sum in (21) by developing $\sigma(x)^{r-1}$ as $\sigma(x)^{r-2}\sum_{j=1}^n x_j^2$ and using the matrices $X(i,j)$. Generally, one can write the following 'inclusion-exclusion' formula for the matrix $\sigma(x)^r M$:

$$\begin{aligned}\sigma(x)^r M = & \sum_{h=1}^{r} \sigma(x)^{r-h} \sum_{\substack{i_1\in V,\ i_2\notin i_1^\perp,\ldots,i_{h-1}\notin i_1^\perp\cup\ldots\cup i_{h-2}^\perp \\ i_h\in V}} x_{i_1}^2\cdots x_{i_h}^2 X(i_1,\ldots,i_h) \\ & + \sum_{h=2}^{r} \sigma(x)^{r-h} \sum_{\substack{i_1\in V,\ i_2\notin i_1^\perp,\ldots,i_{h-1}\notin i_1^\perp\cup\ldots\cup i_{h-2}^\perp \\ i_h\in i_1^\perp\cup\ldots\cup i_{h-1}^\perp}} x_{i_1}^2\cdots x_{i_h}^2 M((i_1,\ldots,i_h)) \\ & + \sum_{\substack{i_1\in V,\ i_2\notin i_1^\perp,\ldots,i_{r-1}\notin i_1^\perp\cup\ldots\cup i_{r-2}^\perp \\ i_r\notin i_1^\perp\cup\ldots\cup i_{r-1}^\perp}} x_{i_1}^2\cdots x_{i_r}^2 M((i_1,\ldots,i_r)).\end{aligned} \tag{24}$$

Therefore, in order to show that $M \in \mathcal{K}^{(r)}_n$, it suffices to show that

$$\begin{gathered} M((i_1,\ldots,i_k,i_{k+1})) \in \mathcal{K}^{(0)}_n \text{ for } S := \{i_1,\ldots,i_k\} \text{ stable}, \\ i_{k+1}\in S^\perp,\ 1\le k\le r-1, \end{gathered} \tag{25}$$

and

$$M((i_1,\ldots,i_r)) \in \mathcal{K}^{(0)}_n \text{ for } \{i_1,\ldots,i_r\} \text{ stable}. \tag{26}$$

For this we need to study the structure of the matrices $M((S))$.

### 3.3 The Structure of the Matrices $M((S))$

Given an ordered stable set $(S) = (i_1, i_2, ..., i_k)$ with $k = 1,\ldots,r$, consider the matrix $M((S))$ from (20) and write

$$M((S)) := \begin{matrix} & S^\perp & V\setminus S^\perp \\ S^\perp & C_k(S) & D_k(S) \\ V\setminus S^\perp & D_k(S)^T & E_k(S) \end{matrix}. \tag{27}$$

**Lemma 4.** *The matrix $M((S))$ from (27) has the following properties.*

*(i) $C_k(S)$ is a $k\times k$ block matrix whose rows and columns are indexed by the partition of $S^\perp$ into $i_1^\perp \cup (i_2^\perp \setminus i_1^\perp)\cup\ldots\cup(i_k^\perp\setminus\{i_1,\ldots,i_{k-1}\}^\perp)$. Let $C_k$ be the skeleton of $C_k(S)$ ($C_k$ is a $k\times k$ matrix) and set $d_k := C_k e \in \mathbb{R}^k$. Then,*

$$e^T C_k e = \sum_{h=1}^{k} d_k(h) = (m_k-1)(t-k)^2.$$

*(ii)* *The matrix* $D_k(S)$ *is a* $k \times 1$ *block matrix, with the same partition as above for the set* $S^\perp$ *indexing its rows. Given* $h \in \{1, \ldots, k\}$*, all entries in the* $(h, 1)$*-block take the same value, which is equal to* $-\frac{d_k(h)}{t-k}$.

*(iii)* *For* $u, v \in V \setminus S^\perp$*, the* $(u, v)$*-th entry of* $E_k(S)$ *is equal to* $tm_{k-1} - 1$ *if* $u \simeq v$ *and to* $-1$ *if* $u \not\simeq v$.

*Proof.* The block structure of the matrices $C_k$ and $D_k$ is determined by the construction of the matrix $M((S))$ in (20) and the shape of the matrices $X(.)$ defined in Section 3.1. We show the lemma by induction on $k \geq 1$. For $k = 1$, the matrix $M((S)) = M - X(i_1)$ has the shape given in (22) and the desired properties hold. Assume (i),(ii),(iii) hold for a stable set $S$ of size $k \geq 1$. Let $i \in V \setminus S^\perp$. We show that (i),(ii),(iii) hold for the stable set $S \cup \{i\}$. Let $D'_k(S)$ (resp., $D''_k(S)$) be the submatrices of $D_k(S)$ whose columns are indexed by $i^\perp \setminus S$ (resp., $V \setminus (S \cup i^\perp)$) and with the same row indices as $D_k(S)$. Then $C_{k+1}(S, i)$ and $D_{k+1}(S, i)$ have the following block structure:

$$C_{k+1}(S, i) = \begin{pmatrix} C_k(S) & D'_k(S) + \frac{t-k-1}{2} m_k J \\ D'_k(S)^T + \frac{t-k-1}{2} m_k J^T & (tm_{k-1} - 1)J \end{pmatrix} \tag{28}$$

$$D_{k+1}(S, i) = \begin{pmatrix} D''_k(S) - \frac{1}{2} m_k J \\ (-1 - m_k \frac{k}{2})J \end{pmatrix}, \tag{29}$$

where $J$ denotes the all-ones matrix of appropriate size. By simple calculation one can show that $C_{k+1}(S, i)$ and $D_{k+1}(S, i)$ satisfy the induction hypothesis.

Finally, the $(u, v)$-th entry of the matrix $E_{k+1}(S, i)$ remains the same as in $E_k(S)$, i.e., equal to $-1$, if $u \not\simeq v$ and, for $u \simeq v$, it is equal to $tm_{k-1} - 1 + km_k = (t - k)m_k - 1 + km_k = tm_k - 1$. □

**Corollary 1.** *Let* $S$ *be a stable set of size* $k = 1, \ldots, r$*. Then,*

$$G((S)) := \begin{pmatrix} C_k(S) & D_k(S) \\ D_k(S)^T & (m_k - 1)J \end{pmatrix} \succeq 0 \iff C_k(S) \succeq 0, \tag{30}$$

$$M((S)) = G((S)) + m_k \begin{pmatrix} 0 & 0 \\ 0 & (t - k)(I + A_{G \setminus S^\perp}) - J \end{pmatrix}, \tag{31}$$

$$M((S, i)) = G((S)) \text{ if } i \in S^\perp. \tag{32}$$

*Proof.* By Lemma 4, $C_k(S)$, $D_k(S)$ are block matrices; hence $G((S)) \succeq 0$ if and only if its skeleton $G := \begin{pmatrix} C_k & -\frac{1}{t-k} C_k e \\ -\frac{1}{t-k} e^T C_k & m_k - 1 \end{pmatrix}$ is positive semidefinite. Now, $G \succeq 0 \iff C_k \succeq 0$ since the last column of $G$ is a linear combination of the first $k$ columns; thus (30) holds. Relations (31), (32) follow using the definitions. □

Therefore, (25), (26) hold (and thus $M \in \mathcal{K}_n^{(r)}$) if we can show that $C_k(S) \succeq 0$ for any stable set $S$ of size $k \leq r$. As $C_k(S)$ is a block matrix, it suffices to show that its skeleton $C_k$ is positive semidefinite. Moreover, it suffices to show that $C_r \succeq 0$ since, in view of (28), the matrices $C_k$ $(1 \leq k \leq r)$ are in fact the leading principal submatrices of $C_r$.

### 3.4 The Matrix $C_r$ Is Positive Semidefinite When $r \le \min(\alpha(G)-1, 6)$

Recall that the entries of $C_r$ depend on the parameter $t$; thus one may alternatively write $C_r$ as $C_r(t)$. Our task is now to show that $C_r(t) \succeq 0$ for all $t \ge r+1$ and $r \le \min(\alpha(G)-1, 6)$. We achieve this by proving that

$$\det C_k(t) > 0 \text{ for } t \ge r+1,\ k = 1, \dots, r. \tag{33}$$

The proof for (33) relies on establishing a recurrence relationship among the determinants of $C_k(t)$. We need the following lemma.

**Lemma 5.** *Assume $C_{k+1}$ is nonsingular. Then,*

$$e^T (C_{k+1})^{-1} e = \frac{t^2}{(t-k)^2} \frac{\det C_k}{\det C_{k+1}}. \tag{34}$$

*Proof.* Write $C_{k+1} := \begin{pmatrix} C_k & x \\ x^T & a \end{pmatrix}$, $(C_{k+1})^{-1} := \begin{pmatrix} A & y \\ y^T & b \end{pmatrix}$. Then,

$$\text{(a) } AC_k + yx^T = I;\ \text{(b) } C_k y + bx = 0;\ \text{(c) } Ax + ay = 0;\ \text{(d) } x^T y + ab = 1. \tag{35}$$

By Lemma 4 and (28), $a = tm_{k-1} - 1 = (t-k)m_k - 1$ and $x = \rho_k e - \frac{1}{t-k} C_k e$, setting $\rho_k := m_k \frac{t-k-1}{2}$. Moreover, $e^T C_k e = (m_k - 1)(t-k)^2$, implying

$$e^T x = k\rho_k - (t-k)(m_k - 1), \quad \frac{e^T x}{t-k} + a = \rho_k \left( \frac{k}{t-k} + 2 \right). \tag{36}$$

Taking the inner product of relation (c) with the all-ones vector and using (35)(a) and (36), we find:

$$\begin{aligned} 0 &= e^T Ax + ae^T y = e^T A(\rho_k e - \tfrac{1}{t-k} C_k e) + ae^T y \\ &= \rho_k e^T Ae - \tfrac{1}{t-k} e^T (I - yx^T) e + ae^T y = \rho_k e^T Ae - \tfrac{k}{t-k} + e^T y (\tfrac{x^T e}{t-k} + a) \\ &= \rho_k (e^T Ae + 2e^T y) + \tfrac{k}{t-k} (\rho_k e^T y - 1); \end{aligned}$$

that is,

$$e^T Ae + 2e^T y = \frac{k}{t-k} \left( \frac{1}{\rho_k} - e^T y \right). \tag{37}$$

Using relations (35)(d),(b) and (36), we find:

$$\begin{aligned} 1 &= x^T y + ab = (\rho_k e - \tfrac{1}{t-k} C_k e)^T y + ab \\ &= \rho_k e^T y + \tfrac{b}{t-k} e^T x + ab = \rho_k e^T y + b\rho_k (\tfrac{k}{t-k} + 2); \end{aligned}$$

that is,

$$e^T y = \frac{1}{\rho_k} - b \left( \frac{k}{t-k} + 2 \right). \tag{38}$$

Relations (37) and (38) imply that $e^T (C_{k+1})^{-1} e = e^T Ae + 2e^T y + b = b \frac{t^2}{(t-k)^2}$. By the cofactor rule, $b = \frac{\det C_k}{\det C_{k+1}}$, and the lemma follows. □

**Corollary 2.** *Let $k \geq 2$ and assume that $C_k(t)$ is nonsingular. Then,*

$$\det C_{k+1}(t) = \frac{2t\rho_k}{t-k} \det C_k(t) - \frac{t^2\rho_k^2}{(t-k+1)^2} \det C_{k-1}(t), \tag{39}$$

*after setting $\rho_k := m_k \frac{t-k-1}{2}$.*

*Proof.* Setting $P := \begin{pmatrix} I & -\frac{1}{t-k}e \\ 0 & 1 \end{pmatrix}$, we find that $P^T C_{k+1} P = \begin{pmatrix} C_k & \rho_k e \\ \rho_k e^T & \mu \end{pmatrix}$, after setting $\mu := m_k \frac{t(t-k-1)}{t-k}$. Set $u := (C_k)^{-1}e$ and let $v_1, \ldots, v_{k+1}$ denote the columns of $P^T C_{k+1} P$. Then, $v_{k+1} - \rho_k(\sum_{i=1}^k u_i v_i)$ has all zero entries except the last $(k+1)$-th entry equal to $\mu - \rho_k^2(\sum_{i=1}^k u_i) = m_k \frac{t(t-k-1)}{t-k} - \rho_k^2 e^T (C_k)^{-1} e$. Therefore, we can conclude that

$$\det C_{k+1} = \det P^T C_{k+1} P = \left( \frac{2t\rho_k}{t-k} - \rho_k^2 e^T (C_k)^{-1} e \right) \det C_k. \tag{40}$$

Relation (39) now follows directly from Lemma 5 and (40). □

**Lemma 6.** *Consider the rational functions $f_1(t) = t-1$, $f_2(t) := \frac{t^2(t-2)(3t-2)}{4(t-1)^2}$ and, for $h = 2, \ldots, k$,*

$$f_{h+1}(t) = \frac{2t\rho_h}{t-h} f_h(t) - \frac{t^2\rho_h^2}{(t-h+1)^2} f_{h-1}(t),$$

*and the polynomials $g_1(t) := 1$, $g_2(t) := 3t-2$ and, for $h = 2, \ldots, k$,*

$$g_{h+1}(t) = \epsilon_h(t-h)g_h(t) - t(t-h-1)g_{h-1}(t),$$

*with $\epsilon_h = 1$ if $h$ is even and $\epsilon_h = 4$ otherwise. As before, $\rho_h := m_h \frac{t-h-1}{2}$.*

*(i) For $h = 2, \ldots, k+1$, $f_h(t) = \frac{t^{\binom{h+1}{2}-1}(t-h)}{4^{\lfloor h/2 \rfloor}(t-1)^h(t-2)^{h-1}\cdots(t-h+1)^2} g_h(t)$.*
*(ii) For $1 \leq k \leq 6$, $g_k(t) > 0$ for all $t \geq k$. Moreover, $g_7(8) > 0$.*

*Proof.* The proof for (i) is by induction on $k$. For (ii), setting $G_k(t) := g_k(t+k)$, one has to show that $G_k(t) > 0$ for $t \geq 0$, $k \leq 6$. This follows from the fact that $G_2(t) = 4 + 3t$, $G_3(t) = 7 + 7t + 2t^2$, $G_4(t) = 64 + 68t + 30t^2 + 5t^3$, $G_5(t) = 167 + 165t + 84t^2 + 25t^3 + 3t^4$, $G_6(t) = 1776 + 1296t + 540t^2 + 248t^3 + 70t^4 + 7t^5$. Moreover, $g_7(8) = 1024$. □

We can now conclude the proof of Theorem 1. Consider $1 \leq r \leq \min(\alpha(G), 6)$ and $t \geq r+1$. We show that (33) holds using Corollary 2 and Lemma 6. First note that $\det C_h(t) = f_h(t)$ for $h = 1, 2$ (direct verification). Let $k \in \{1, \ldots, r\}$. If $k = 1, 2$, then $\det C_k(t) > 0$. Assume $k \geq 3$ and $C_{k-1}(t) \succ 0$. By Corollary 2, $\det C_1(t), \ldots, \det C_k(t)$ are related via (39); that is, $\det C_h(t) = f_h(t)$ for $h = 1, \ldots, k$. We now deduce from Lemma 6 that $\det C_k(t) > 0$. This shows that $C_r(t) \succ 0$ for $t \geq r+1$, which concludes the proof of Theorem 1.

Let us now conclude the proof of Theorem 2 in the case when $\alpha(G) = 8$. We have to show that the matrix $M = t(I + A_G) - J$ from (18) with $t := \alpha(G) = 8$ belongs to $\mathcal{K}_n^{(7)}$. We use the same argument as in the proof of Theorem 1. Thus we are left with the task of proving that $\det C_1(t), \ldots, \det C_7(t) > 0$ for $t = 8$. This follows from the assertions $g_1(8), \ldots, g_6(8), g_7(8) > 0$ in Lemma 6.

Note that the same argument cannot be used for proving Conjecture 1 in the case $\alpha(G) = 9$, since $g_1(9), \ldots, g_6(9) > 0$ while $g_7(9) < 0$ which implies that the matrix $C_7(9)$ is not positive semidefinite.

## 4 Proof of Theorem 3

Obviously, $las^{(1)} = \vartheta^{(0)}(G)$. In view of (14), we have to show that $las^{(r)} \leq \tilde{\vartheta}^{(r-1)}(G)$ for any positive integer $r$. For this, let $x \in \mathbb{R}^{\mathcal{P}_{2r}(V)}$ be feasible for (1), i.e., $x_\emptyset = 1$, $x_I \geq 0$ ($|I| = r+1$), $x_{ij} = 0$ ($ij \in E$), and $M_r(x) \succeq 0$. Then, $x_I = 0$ for any $I \in \mathcal{P}_{2r}(V)$ containing an edge. We may assume that $\sum_{i=1}^n x_i > 0$. For $p = 1, \ldots, r+1$, define

$$\ell_p := \sum_{\beta \in I(n,p-1)} \frac{(p-1)!}{\beta!} x_{S(\beta)}.$$

Then, $\ell_1 = 1$, $\ell_p \geq \ell_2 = \sum_{i=1}^n x_i > 0$ for $p \geq 2$. For $p = 1, \ldots, r$, define $y = (y_\delta)_{\delta \in I(n,2p+2)}$ as follows: $y_\delta = 0$ if $S_{odd}(\delta) \neq \emptyset$, $y_\delta := \frac{1}{\ell_p} x_{S(\delta)}$ otherwise (then $|S(\delta)| \leq p+1 \leq r+1$).

**Lemma 7.** $N_{p+1}(y) \succeq 0$.

*Proof.* For $I \subseteq V$, set $\mathcal{O}_I := \{\beta \in I(n,p+1) \mid S_{odd}(\beta) = I\}$ and $N_I := (y_{\beta+\beta'})_{\beta,\beta' \in \mathcal{O}_I}$. Then, $N_{p+1}(y)$ is a block diagonal matrix with the matrices $N_I$ ($I \subseteq V$) as diagonal blocks. As $\ell_p N_I = (x_{S(\beta) \cup S(\beta')})_{\beta,\beta' \in \mathcal{O}_I}$, $N_I \succeq 0$ since it is obtained from a principal submatrix of $M_r(x)$ by duplicating certain rows/columns (unless $|I| = r+1$ in which case $N_I$ is the $1 \times 1$ matrix with entry $x_{|I|} \geq 0$, implying again $N_I \succeq 0$). □

Therefore, the matrix $Z(p) := C(y) = \sum_{\gamma \in I(n,p-1)} \frac{(p-1)!}{\gamma!} N^\gamma(y)$ belongs to the cone $\mathcal{C}_n^{(p-1)}$. Moreover, $Z(p)_{ij} = 0$ if $ij \in E$. Define the matrix

$$\tilde{Z}(p) := \begin{pmatrix} 1 & Z(p)_{11} \ldots Z(p)_{nn} \\ Z(p)_{11} & \\ \vdots & Z(p) \\ Z(p)_{nn} & \end{pmatrix}. \tag{41}$$

**Lemma 8.** $\tilde{Z}(p) \succeq 0$.

*Proof.* The matrix:

$$\ell_p \tilde{Z}(p) = \sum_{\gamma \in I(n,p-1)} \frac{(p-1)!}{\gamma!} \begin{pmatrix} x_{S(\gamma)} & y_{2\gamma+4e_1} \cdots y_{2\gamma+4e_n} \\ y_{2\gamma+4e_1} & \\ \vdots & (y_{2\gamma+2e_j+2e_k})_{j,k=1}^n \\ y_{2\gamma+4e_n} & \end{pmatrix}$$
$$= \sum_{\gamma \in I(n,p-1)} \frac{(p-1)!}{\gamma!} \begin{pmatrix} x_{S(\gamma)} & x_{S(\gamma+e_1)} \cdots x_{S(\gamma+e_n)} \\ x_{S(\gamma+e_1)} & \\ \vdots & (x_{S(\gamma+e_j+e_k)})_{j,k=1}^n \\ x_{S(\gamma+e_n)} & \end{pmatrix}$$

is positive semidefinite, since the matrices in the above summation are principal submatrices of $M_r(x)$. □

**Lemma 9.** $\sum_{i,j=1}^n Z(p)_{ij} = \frac{\ell_{p+2}}{\ell_p}$ *and* $\sum_{i=1}^n Z(p)_{ii} = \frac{\ell_{p+1}}{\ell_p}$.

*Proof.* Direct verification. □

**Lemma 10.** $\frac{\ell_{p+2}}{\ell_{p+1}} \geq \frac{\ell_{p+1}}{\ell_p}$.

*Proof.* By Lemma 8, $\tilde{Z}(p) \succeq 0$, implying $Z(p) - diag(Z(p))diag(Z(p))^T \succeq 0$. Therefore, $e^T(Z(p) - diag(Z(p))diag(Z(p))^T)e \geq 0$, yielding $\sum_{i,j=1}^n Z(p)_{ij} \geq (\sum_{i=1}^n Z(p)_{ii})^2$. The result now follows using Lemma 9. □

From Lemmas 9 and 10, we deduce that $\sum_{i=1}^n Z(r)_{ii} = \frac{\ell_{r+1}}{\ell_r} \geq \frac{\ell_2}{\ell_1} = \sum_{i=1}^n x_i$. As the matrix $Z(r)$ is feasible for the program (13) defining the parameter $\tilde{\vartheta}^{(r)}(G)$, this shows that $\tilde{\vartheta}^{(r)}(G) \geq \sum_{i=1}^n x_i$ and thus $\tilde{\vartheta}^{(r)}(G) \geq las^{(r)}(G)$, concluding the proof of Theorem 3.

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